# A GENERAL NUMERICAL METHOD FOR THE SOLUTION OF GRAVITY WAVE PROBLEMS. PART 1: 2D STEEP GRAVITY WAVES IN SHALLOW WATER

S. J. LIAO

*Institute of Shipbuilding, University of Hamburg, Laemmersieth 90, D-2000 Hamburg 60, F.R.G.*

## SUMMARY

In this paper, 2D steep gravity waves in shallow water are used to introduce and examine a new kind of numerical method for the solution of non-linear problems called the finite process method (FPM). On the basis of the velocity potential function and the FPM, a numerical method for 2D non-linear gravity waves in shallow water is described which can be applied to solve 3D problems, e.g. the wave resistance of a ship moving in deep or shallow water. The convergence is examined and a comparison with the results of other authors is made.

The FPM can successfully avoid the use of iterative methods and therefore can overcome the disadvantages and limitations of such methods. In contrast to iterative methods, the FPM is insensitive to the selection of the initial solution and the number of unknowns. The basic idea of the FPM can be used to solve other non-linear problems. Its disadvantage is that much more CPU time is needed to obtain a sufficiently accurate result.

KEY WORDS   Non-linear gravity waves   Velocity potential   Transformation from non-linear to linear

## 1. INTRODUCTION

In the study of gravity waves it is generally assumed that the fluid is inviscid. If the flow is irrotational, there exists a velocity potential function $\Phi(x, y, z; t)$ from which the velocity components $(u, v, w)$ can be obtained. If the fluid is also incompressible, the velocity potential function $\Phi(x, y, z; t)$ satisfies the Laplace equation

$$\nabla^2 \Phi = 0$$

throughout the fluid.

Although the above governing equation for $\Phi$ is a linear Laplace equation, its two boundary conditions on the free surface, called the dynamic and kinematic conditions of the free surface, are unfortunately non-linear and must be satisfied on the unknown free surface. Therefore gravity wave problems are generally non-linear. This non-linearity leads to difficulties in the numerical computation of these types of problems. A particular difficulty for the solution of gravity waves, which is perhaps the main difficulty, is that the free surface elevation is unknown but the two boundary conditions must be satisfied on it.

Generally, a non-linear discretized algebraic equation with a large number of unknown quantities will be obtained after numerical discretization of a non-linear problem. The general methods used widely to solve these kinds of non-linear algebraic equations are iterative methods. However, the convergence of iterative methods depends greatly on not only the selection of initial

values but also the number of unknown quantities, i.e. the convergence is sensitive to the number of unknowns. Another approach is to use semidirect methods[14] which can partly overcome the limitation of iterative methods. However, semidirect methods still use the iterative technique to solve a set of non-linear algebraic equations at each step of the computation, i.e. semidirect methods cannot avoid the application of iterative methods.

In this paper we try to use 2D gravity waves in shallow water as an example to describe a new kind of numerical method for general non-linear problems called the finite process method (FPM). The FPM avoids the use of iterative methods. From this viewpoint, one could call it a type of direct method for the solution of non-linear problems.

In 1979, Vanden-Broeck and Schwartz[1] reported on the numerical computation of 2D non-linear gravity waves in shallow water. They used an analytical function of a complex variable $z$ to represent the solution and obtained a non-linear integrodifferential equation for the free surface which was solved by Newton's iterative technique. In 1981, Rienecker and Fenton[2] researched the same problem from a different angle. They applied a finite Fourier series to give a set of non-linear equations which were also solved by using Newton's iterative method. Many other authors (e.g. References 3–11) have studied 2D gravity waves experimentally, theoretically or numerically. The numerical results given by different authors using different methods agree well with the theoretical results and even with the experimental results.

The works described above are excellent. If looked at solely from the viewpoint of the numerical computation of 2D gravity waves in shallow water, the present work described in this paper would be of little relevance. However, it should be emphasized that the authors mentioned above used Newton's iterative method to solve the non-linear algebraic equations. On the other hand, their method could not be applied to solve 3D gravity wave problems, e.g. to compute the wave resistance of a ship moving with a constant velocity in shallow water.

The main purpose of this paper is not to discuss more deeply the numerical method for 2D gravity waves in shallow water, but to describe a new kind of numerical method which avoids the use of iterative methods. Another purpose of this paper is to describe a general method for the solution of 2D and 3D non-linear gravity wave problems which is based on the velocity potential function.

## 2. MATHEMATICAL DESCRIPTION

The problem considered here is that of 2D periodic waves moving without change of form over a layer of fluid on a horizontal bed. With horizontal co-ordinate $x$ and vertical co-ordinate $y$, the origin is on the bed and moves with the same velocity $C$ as the waves. Therefore all motion is steady in the frame of reference. Let $D$ denote the average depth of the water.

Suppose the flow is irrotational. Then there exists a velocity potential function $\phi(x, y)$ such that the velocity components $(u, v)$ are given by

$$u = \frac{\partial \phi}{\partial x}, \qquad v = \frac{\partial \phi}{\partial y}.$$

Suppose the fluid is incompressible. Then $\phi(x, y)$ satisfies Laplace's equation in the field:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad \text{in } \Omega.$$

Suppose the fluid is inviscid and without surface tension. Then, on the free surface, $\phi(x, y)$ satisfies

$$C^2 \phi_{xx} + g\phi_y + \tfrac{1}{2}\nabla\phi\nabla(\nabla\phi\nabla\phi) - 2C\nabla\phi\nabla\phi_x = 0 \quad \text{on } y = \eta(x), \tag{1}$$

where

$$\eta(x) = \frac{1}{g}(C\phi_x - \tfrac{1}{2}\nabla\phi\nabla\phi) \quad \text{on } y = \eta(x). \tag{2}$$

Here $g$ is the gravitational acceleration, $\eta(x)$ denotes the wave elevation and $C$ is the wave velocity.

Expression (2) is the dynamic condition of the free surface. Expression (1) is obtained from the dynamic and kinematic conditions of the free surface, and the relation

$$\frac{\partial}{\partial t} = -C\frac{\partial}{\partial x}$$

is used.

On the bed, $\phi(x, y)$ satisfies

$$\frac{\partial\phi}{\partial y} = 0 \quad \text{on } y = 0. \tag{3}$$

For simplicity, define

$$
\begin{aligned}
R[\phi(x,y)] &= C^2\phi_{xx} + g\phi_y + \tfrac{1}{2}\nabla\phi\nabla(\nabla\phi\nabla\phi) - 2C\nabla\phi\nabla\phi_x \\
&= C^2\phi_{xx} + g\phi_y + (\phi_x)^2\phi_{xx} + 2\phi_x\phi_y\phi_{xy} + (\phi_y)^2\phi_{yy} - 2C(\phi_x\phi_{xx} + \phi_y\phi_{yx}) \\
&\quad \text{on } y = \eta(x),
\end{aligned}
\tag{4}
$$

$$
\begin{aligned}
H[\phi(x,y)] &= \frac{1}{g}(C\phi_x - \tfrac{1}{2}\nabla\phi\nabla\phi) \\
&= \frac{1}{g}[C\phi_x - \tfrac{1}{2}(\phi_x^2 + \phi_y^2)] \quad \text{on } y = \eta(x).
\end{aligned}
\tag{5}
$$

Then the problem considered can be described as

$$\nabla^2\phi(x, y) = 0 \quad \text{in } \Omega, \tag{6}$$

with boundary conditions

$$R[\phi(x, y)] = 0 \quad \text{on } y = \eta(x), \tag{7}$$

$$\frac{\partial\phi}{\partial y} = 0 \quad \text{on } y = 0, \tag{8}$$

where

$$\eta(x) = H[\phi(x, y)] \quad \text{on } y = \eta(x). \tag{9}$$

## 3. BASIC IDEA OF FINITE PROCESS METHOD

Let $f_1(p)$ and $f_2(p)$ be real continuous functions which satisfy

$$f_1(p) = \begin{cases} 0 & \text{for } p = 0, \\ 1 & \text{for } p = 1, \end{cases} \qquad f_2(p) = \begin{cases} 1 & \text{for } p = 0, \\ 0 & \text{for } p = 1, \end{cases}$$

where $f_1(p)$ could be called the first type of process function and $f_2(p)$ the second type of process function. The simplest first-type process function is $f_1(p) = p$ and the simplest second-type process function is $f_2(p) = 1 - p$.

Then a kind of continuous mapping $\phi(x, y) \to \phi(x, y; p)$, $\eta(x) \to \eta(x; p)$, $\Omega \to \Omega(p)$ can be constructed as follows:

$$\nabla^2 \phi(x, y; p) = 0 \quad \text{in } \Omega(p),\tag{10}$$

with boundary conditions

$$f_1(p)R[\phi(x, y; p)] + f_2(p)\{R[\phi(x, y; p)] - R[\phi_0(x, y)]\} = 0 \quad \text{on } y = \eta(x; p),\tag{11}$$

$$\frac{\partial \phi(x, y; p)}{\partial y} = 0 \quad \text{on } y = 0,\tag{12}$$

where

$$\eta(x; p) = f_1(p)H[\phi(x, y; p)] + f_2(p)\eta_0(x) \quad \text{on } y = \eta(x; p).\tag{13}$$

Here $\eta_0(x)$ is a freely selected initial wave elevation; $\phi_0$ is a freely selected initial velocity potential which should satisfy

$$\nabla^2 \phi_0(x, y) = 0 \quad \text{in } \Omega_0,\tag{14}$$

with boundary condition

$$\frac{\partial \phi_0}{\partial y} = 0 \quad \text{on } y = 0,\tag{15}$$

where the initial definition domain $\Omega_0$ of the fluid field is defined by $y = \eta_0(x)$ and $y = 0$.

For simplicity, $\phi(x, y; p)$ and $\eta(x; p)$, the mapping of $\phi(x, y)$ and $\eta(x)$, could be called the zero-order process; thus (10)–(13) could be called the zero-order process equations.

When $p = 0$, from the zero-order process equations (10)–(13) one can obtain the initial equation

$$\nabla^2 \phi(x, y; 0) = 0 \quad \text{in } \Omega_0,\tag{16}$$

with boundary conditions

$$R(\phi) = R(\phi_0) \quad \text{on } y = \eta_0(x),\tag{17}$$

$$\frac{\partial \phi}{\partial y} = 0 \qquad \text{on } y = 0,\tag{18}$$

while

$$\eta(x; 0) = \eta_0(x).\tag{19}$$

One can see that the initial velocity potential $\phi_0(x, y)$ which satisfies the linear equations (14) and (15) is just the solution of the non-linear initial equations (16)–(19), i.e.

$$\phi(x, y; 0) = \phi_0(x, y).$$

When $p = 1$, from the zero-order process equations (10)–(13) one can obtain the final equation

$$\nabla^2 \phi(x, y; 1) = 0 \quad \text{in } \Omega_f,\tag{20}$$

with boundary conditions

$$R[\phi(x, y; 1)] = 0 \quad \text{on } y = \eta(x; 1),\tag{21}$$

$$\frac{\partial \phi(x, y; 1)}{\partial y} = 0 \quad \text{on } y = 0,\tag{22}$$

while

$$\eta(x; 1) = H[\phi(x, y; 1)] \quad \text{on} \quad y = \eta(x; 1). \tag{23}$$

The final equations (20)–(23) are the same as the original equations (6)–(9) described in Section 2, which are just what one wants to solve. Here $\phi(x, y; 1)$ could be called the final velocity potential, denoted as $\phi_f(x, y)$; and $\eta(x; 1)$ could be called the final wave elevation, denoted as $\eta_f(x)$.

From the above analysis one can see that the zero-order process equations (10)–(13) connect the freely selected initial velocity potential $\phi_0(x, y)$ and initial wave elevation $\eta_0(x)$ with solutions of the original equations (6)–(9) described in Section 2. The process of changing the variable $p$ from zero to unity is just the process of transformation from the selected initial solution to the final solution. The zero-order process equations (10)–(13) are a special continuous mapping which gives a kind of relation between the selected initial solution $\phi_0(x, y)$, $\eta_0(x)$ and the final solution $\phi_f(x, y)$, $\eta_f(x)$. However, this kind of relation is also non-linear in non-linear problems. In the following part of this section we try to give a linear relation between initial solution $\phi_0(x, y)$, $\eta_0(x)$ and final solution $\phi_f(x, y)$, $\eta_f(x)$ by using the concept of *process derivatives*.

Define

$$\phi^{[1]}(x, y; p) = \frac{\partial \phi(x, y; p)}{\partial p}, \tag{24}$$

$$\eta^{[1]}(x; p) = \frac{\partial \eta(x; p)}{\partial p} \tag{25}$$

as the first-order process derivatives of $\phi(x, y; p)$ and $\eta(x; p)$. Then the freely selected initial solution $\phi_0(x, y)$, $\eta_0(x)$ and the final solution $\phi_f(x, y)$, $\eta_f(x)$ have the following integral relations:

$$\phi_f(x, y) = \phi(x, y; 1)$$

$$= \phi(x, y; 0) + \int_0^1 \phi^{[1]}(x, y; p) \, dp$$

$$= \phi_0(x, y) + \int_0^1 \phi^{[1]}(x, y; p) \, dp, \tag{26}$$

$$\eta_f(x) = \eta(x; 1)$$

$$= \eta(x; 0) + \int_0^1 \eta^{[1]}(x; p) \, dp$$

$$= \eta_0(x) + \int_0^1 \eta^{[1]}(x; p) \, dp. \tag{27}$$

The equations for $\phi^{[1]}(x, y; p)$ and $\eta^{[1]}(x; p)$ can be obtained from the zero-order process equations (10)–(13) in the following way.

From (10) and (12) one has

$$\frac{\partial}{\partial p}(\nabla^2 \phi) = \nabla^2 \phi^{[1]}(x, y; p) = 0 \quad \text{in } \Omega(p), \tag{28}$$

$$\frac{\partial}{\partial p}\left(\frac{\partial \phi}{\partial y}\right) = \frac{\partial \phi^{[1]}(x, y; p)}{\partial y} = 0 \quad \text{on } y = 0. \tag{29}$$

The dynamic and kinematic conditions of the free surface must be satisfied on the wave elevation $y = \eta(x; p)$, which is also a function of the process-independent variable $p$. Therefore deriving (11)

and (13) for the process-independent variable $p$ one can obtain

$$f_1'(p)R[\phi(x, y; p)] + f_2'(p)\{R[\phi(x, y; p)] - R[\phi_0(x, y)]\} + [f_1(p) + f_2(p)]\frac{dR[\phi(x, y; p)]}{dp} = 0$$

$$\text{on } y = \eta(x; p), \quad (30)$$

$$\eta^{[1]}(x; p) = f_1'(p)H[\phi(x, y; p)] + f_1(p)\frac{dH[\phi(x, y; p)]}{dp} + f_2'(p)\eta_0(x). \quad (31)$$

On the wave elevation $y = \eta(x; p)$ one has

$$\frac{dR}{dp} = \frac{\partial R}{\partial p} + \eta^{[1]}\frac{\partial R}{\partial y} \quad \text{on } y = \eta(x; p), \quad (32)$$

$$\frac{dH}{dp} = \frac{\partial H}{\partial p} + \eta^{[1]}\frac{\partial H}{\partial y} \quad \text{on } y = \eta(x; p). \quad (33)$$

Substituting (33) in (31) one has

$$\eta^{[1]}(x; p) = \frac{f_1'(p)H[\phi(x, y; p)] + f_2'(p)\eta_0(x) + f_1(p)\,\partial H[\phi(x, y; p)]/\partial p}{1 - f_1(p)\partial H[\phi(x, y; p)]/\partial y} \quad \text{on } y = \eta(x; p). \quad (34)$$

For simplicity, define

$$S[\phi(x, y; p); p] = \frac{\partial R[\phi(x, y; p)]/\partial y}{1 - f_1(p)\partial H[\phi(x, y; p)]/\partial y} \quad \text{on } y = \eta(x; p). \quad (35)$$

Substituting (32) in (30) and using (34) and (35) one has

$$\frac{\partial R[\phi(x, y; p)]}{\partial p} + f_1(p)S[\phi(x, y; p); p]\frac{\partial H[\phi(x, y; p)]}{\partial p} = T[\phi(x, y; p); p] \quad \text{on } y = \eta(x; p), \quad (36)$$

where

$$T[\phi(x, y; p); p] = -\frac{[f_1'(p) + f_2'(p)]R[\phi(x, y; p)] - f_2'(p)R[\phi_0(x, y)]}{f_1(p) + f_2(p)}$$

$$- S[\phi(x, y; p); p]\{f_1'(p)H[\phi(x, y; p)] + f_2'(p)\eta_0(x)\} \quad \text{on } y = \eta(x; p). \quad (37)$$

According to (4) one has

$$\frac{\partial R}{\partial p} = \frac{\partial R}{\partial \phi_x}\frac{\partial \phi_x}{\partial p} + \frac{\partial R}{\partial \phi_y}\frac{\partial \phi_y}{\partial p} + \frac{\partial R}{\partial \phi_{xx}}\frac{\partial \phi_{xx}}{\partial p} + \frac{\partial R}{\partial \phi_{xy}}\frac{\partial \phi_{xy}}{\partial p} + \frac{\partial R}{\partial \phi_{yy}}\frac{\partial \phi_{yy}}{\partial p}$$

$$= \frac{\partial R}{\partial \phi_x}\phi_x^{[1]} + \frac{\partial R}{\partial \phi_y}\phi_y^{[1]} + \frac{\partial R}{\partial \phi_{xx}}\phi_{xx}^{[1]} + \frac{\partial R}{\partial \phi_{xy}}\phi_{xy}^{[1]} + \frac{\partial R}{\partial \phi_{yy}}\phi_{yy}^{[1]} \quad \text{on } y = \eta(x; p). \quad (38)$$

In the same way, according to (5) one can obtain

$$\frac{\partial H}{\partial p} = \frac{\partial H}{\partial \phi_x}\frac{\partial \phi_x}{\partial p} + \frac{\partial H}{\partial \phi_y}\frac{\partial \phi_y}{\partial p}$$

$$= \frac{\partial H}{\partial \phi_x}\phi_x^{[1]} + \frac{\partial H}{\partial \phi_y}\phi_y^{[1]} \quad \text{on } y = \eta(x; p). \quad (39)$$

Thus substituting (38) and (39) in (36) one has

$$\frac{\partial R[\phi(x, y; p)]}{\partial p} + f_1(p)S[\phi(x, y; p); p]\frac{\partial H[\phi(x, y; p)]}{\partial p}$$

$$= \left(\frac{\partial R}{\partial \phi_x} + f_1(p)S[\phi(x, y; p); p]\frac{\partial H}{\partial \phi_x}\right)\phi_x^{[1]} + \left(\frac{\partial R}{\partial \phi_y} + f_1(p)S[\phi(x, y; p); p]\frac{\partial H}{\partial \phi_y}\right)\phi_y^{[1]}$$

$$+ \left(\frac{\partial R}{\partial \phi_{xx}} - \frac{\partial R}{\partial \phi_{yy}}\right)\phi_{xx}^{[1]} + \frac{\partial R}{\partial \phi_{xy}}\phi_{xy}^{[1]}$$

$$= T[\phi(x, y; p); p] \quad \text{on } y = \eta(x; p). \tag{40}$$

Here the relation

$$\phi_{xx}^{[1]} + \phi_{yy}^{[1]} = 0 \quad \text{in } \Omega(p) \tag{41}$$

has been used.

Therefore, according to (28), (29), (34) and (40), $\phi^{[1]}(x, y; p)$ and $\eta^{[1]}(x; p)$ satisfy the partial differential equation

$$\nabla^2\phi^{[1]}(x, y; p) = 0 \quad \text{in } \Omega(p), \tag{42}$$

with boundary conditions

$$\left(\frac{\partial R}{\partial \phi_x} + f_1(p)S[\phi(x, y; p); p]\frac{\partial H}{\partial \phi_x}\right)\phi_x^{[1]} + \left(\frac{\partial R}{\partial \phi_y} + f_1(p)S[\phi(x, y; p); p]\frac{\partial H}{\partial \phi_y}\right)\phi_y^{[1]}$$

$$+ \left(\frac{\partial R}{\partial \phi_{xx}} - \frac{\partial R}{\partial \phi_{yy}}\right)\phi_{xx}^{[1]} + \frac{\partial R}{\partial \phi_{xy}}\phi_{xy}^{[1]} = T[\phi(x, y; p); p] \quad \text{on } y = \eta(x; p), \tag{43}$$

$$\frac{\partial\phi^{[1]}(x, y; p)}{\partial y} = 0 \quad \text{on } y = 0, \tag{44}$$

while

$$\eta^{[1]}(x; p) = \frac{f_1'(p)H[\phi(x, y; p)] + f_2'(p)\eta_0(x) + f_1(p)\partial H[\phi(x, y; p)]/\partial p}{1 - f_1(p)\partial H[\phi(x, y; p)]/\partial y} \quad \text{on } y = \eta(x; p). \tag{45}$$

Equations (42)–(45) could be called the first-order process equations.

It is very interesting that equations (42)–(44) are a set of linear partial differential equations for $\phi^{[1]}(x, y; p)$. If $\phi(x, y; p)$ and $\eta(x; p)$ are known, $\phi^{[1]}(x, y; p)$ and $\eta^{[1]}(x; p)$ can be obtained by solving these linear partial differential equations. Then according to (26) and (27) one has

$$\phi(x, y; p + \Delta p) = \phi(x, y; p) + \phi^{[1]}(x, y; p)\Delta p,$$

$$\eta(x; p + \Delta p) = \eta(x; p) + \eta^{[1]}(x; p)\Delta p.$$

In this way one can obtain the final solution $\phi_f(x, y)$, $\eta_f(x)$ indirectly. Generally the first-order process equations are linear although the zero-order process is non-linear. This is the most important property of the process. A pure mathematical proof of this property can be given but will be described in other papers of the author.

If we solve the non-linear zero-order process equations (10)–(13) directly, it seems difficult to avoid using iterative methods. However, by solving the $n_p$ linear first-order process equations (42)–(45) for $\phi^{[1]}(x, y; p)$ and $\eta^{[1]}(x; p)$ ($n_p \geq 1$, $\Delta p = 1/n_p$), we can obtain the solution of the original equations (6)–(9) indirectly without using iterative methods. In other words, a non-linear

problem can be transformed to a complex linear problem by using the concept of process derivatives.

From (4) and (5) one can obtain

$$\frac{\partial R[\phi(x, y; p)]}{\partial \phi_x} = 2[(\phi_x - C)\phi_{xx} + \phi_y \phi_{xy}], \tag{46}$$

$$\frac{\partial R[\phi(x, y; p)]}{\partial \phi_y} = g + 2[(\phi_x - C)\phi_{xy} + \phi_y \phi_{yy}], \tag{47}$$

$$\frac{\partial R[\phi(x, y; p)]}{\partial \phi_{xx}} = (\phi_x - C)^2, \tag{48}$$

$$\frac{\partial R[\phi(x, y; p)]}{\partial \phi_{xy}} = 2(\phi_x - C)\phi_y, \tag{49}$$

$$\frac{\partial R[\phi(x, y; p)]}{\partial \phi_{yy}} = \phi_y^2, \tag{50}$$

$$\frac{\partial H[\phi(x, y; p)]}{\partial \phi_x} = -\frac{1}{g}(\phi_x - C), \tag{51}$$

$$\frac{\partial H[\phi(x, y; p)]}{\partial \phi_y} = -\frac{1}{g}\phi_y, \tag{52}$$

$$\frac{\partial R[\phi(x, y; p)]}{\partial y} = \frac{\partial R}{\partial \phi_x}\phi_{xy} + \frac{\partial R}{\partial \phi_y}\phi_{yy} + \frac{\partial R}{\partial \phi_{xx}}\phi_{xxy} + \frac{\partial R}{\partial \phi_{xy}}\phi_{xyy} + \frac{\partial R}{\partial \phi_{yy}}\phi_{yyy}$$

$$= \frac{\partial R}{\partial \phi_x}\phi_{xy} - \frac{\partial R}{\partial \phi_y}\phi_{xx} + \left(\frac{\partial R}{\partial \phi_{xx}} - \frac{\partial R}{\partial \phi_{yy}}\right)\phi_{xxy} + \frac{\partial R}{\partial \phi_{xy}}\phi_{xyy}, \tag{54}$$

$$\frac{\partial H[\phi(x, y; p)]}{\partial y} = \frac{\partial H}{\partial \phi_x}\phi_{xy} + \frac{\partial H}{\partial \phi_y}\phi_{yy} = \frac{\partial H}{\partial \phi_x}\phi_{xy} - \frac{\partial H}{\partial \phi_y}\phi_{xx}. \tag{55}$$

The process of numerical computation can be described simply as follows.

1. Select the initial wave elevation $\eta_0(x)$ and initial velocity potential $\phi_0(x, y)$.
2. Let $p_i = i\Delta p$ for $i = 0, 1, 2, \ldots, n_p - 1$, where, $\Delta p = 1/n_p$.
3. Solve the linear partial differential equations (42)–(44) to obtain the value of $\phi^{[1]}(x, y; p_i)$.
4. Compute the value of $\eta^{[1]}(x; p_i)$ by means of expression (34).
5. Compute the values of wave elevation and velocity potential at the next step $p_{i+1}$:

$$\phi(x, y; p_{i+1}) = \phi(x, y; p_i) + \phi^{[1]}(x, y; p_i)\Delta p, \tag{56}$$

$$\eta(x; p_{i+1}) = \eta(x; p_i) + \eta^{[1]}(x; p_i)\Delta p. \tag{57}$$

6. If $p_{i+1} < 1$, then go back to step 2.

Here the definition region $p \in [0, 1]$ of the process-independent variable $p$ is discretized, which is why this method is called the finite process method.

One can see that in the above numerical computation, only $n_p$ linear partial differential equations (42)–(44) need be solved ($n_p \geq 1$, $\Delta p = 1/n_p$). This means that the original non-linear problem has been transformed to a set of $n_p$ linear problems ($n_p \geq 1$). Theoretically, by means of a large computer, almost all linear problems can be solved without great difficulty. Thus from the

viewpoint of the FPM, theoretically it seems that a non-linear problem could also be solved by a large computer because the non-linear problem can be approximated by a set of $n_p$ linear problems ($n_p \geqslant 1$). In other words, a non-linear problem is just a very complex linear one from the viewpoint of the FPM.

In the next section, detailed numerical formulae and numerical results for 2D progressive gravity waves in shallow water are given. It should be emphasized that the numerical method described here can also be used to solve 3D gravity wave problems, e.g. the wave resistance of a ship moving in deep or shallow water, by making only a simple expansion and using the Hess–Smith method.

## 4. NUMERICAL SOLUTION OF GRAVITY WAVES IN SHALLOW WATER

### 4.1. Numerical formulae

According to the derivation in the above section, the zero-order process equation for the velocity potential function $\phi(x, y)$ of 2D progressive gravity waves in shallow water can be written as

$$\nabla^2 \phi(x, y; p) = 0 \quad \text{in } \Omega(p), \tag{58}$$

with boundary conditions

$$f_1(p) R[\phi(x, y; p)] + f_2(p)\{R[\phi(x, y; p)] - R[\phi_0(x, y)]\} = 0, \quad \text{on } y = \eta(x; p), \tag{59}$$

$$\frac{\partial \phi(x, y; p)}{\partial y} = 0 \quad \text{on } y = 0, \tag{60}$$

while

$$\eta(x; p) = f_1(p) H(x, y; p) + f_2(p) \eta_0(x) \quad \text{on } y = \eta(x; p). \tag{61}$$

The corresponding first-order process equation is

$$\nabla^2 \phi^{[1]}(x, y; p) = 0 \quad \text{in } \Omega(p), \tag{62}$$

with boundary conditions

$$b_1(x, y; p)\phi_x^{[1]}(x, y; p) + b_2(x, y; p)\phi_y^{[1]}(x, y; p) + b_3(x, y; p)\phi_{xx}^{[1]}(x, y; p) + b_4(x, y; p)\phi_{xy}^{[1]}(x, y; p)$$
$$= T[\phi(x, y; p)] \quad \text{on } y = \eta(x; p), \tag{63}$$

$$\frac{\partial \phi^{[1]}(x, y; p)}{\partial y} = 0 \quad \text{on } y = 0, \tag{64}$$

where

$$b_1(x, y; p) = \frac{\partial R[\phi(x, y; p)]}{\partial \phi_x} + f_1(p) S[\phi(x, y; p); p] \frac{\partial H[\phi(x, y; p)]}{\partial \phi_x} \quad \text{on } y = \eta(x; p), \tag{65}$$

$$b_2(x, y; p) = \frac{\partial R[\phi(x, y; p)]}{\partial \phi_y} + f_1(p) S[\phi(x, y; p); p] \frac{\partial H[\phi(x, y; p)]}{\partial \phi_y} \quad \text{on } y = \eta(x; p), \tag{66}$$

$$b_3(x, y; p) = \frac{\partial R[\phi(x, y; p)]}{\partial \phi_{xx}} - \frac{\partial R[\phi(x, y; p)]}{\partial \phi_{yy}} \quad \text{on } y = \eta(x; p), \tag{67}$$

$$b_4(x, y; p) = \frac{\partial R[\phi(x, y; p)]}{\partial \phi_{xy}} \quad \text{on } y = \eta(x; p), \tag{68}$$

while $\eta^{[1]}(x; p)$ satisfies

$$\eta^{[1]}(x; p) = \frac{f_1'(p)H[\phi(x, y; p)] + f_2'(p)\eta_0(x) + f_1(p)\partial H[\phi(x, y; p)]/\partial p}{1 - f_1(p)\partial H[\phi(x, y; p)]/\partial y} \quad \text{on } y = \eta(x; p). \quad (69)$$

Let $k = 2\pi/L_w$ denote the wave number. Then it is easily seen that

$$\phi(x, y; p) = \sum_{j=1}^{\infty} a_j(p)\phi_j(x, y), \quad (70)$$

with

$$\phi_j(x, y) = \frac{\cosh(jky)\sin(jkx)}{\cosh(jkD)}, \quad (71)$$

satisfies the Laplace equation (58) and the bed condition (60). Here the $a_j(p)$ $(j = 1, 2, \dots)$ are only dependent on the process-independent variable $p$.

Then from (70) one has

$$\frac{\partial\phi(x, y; p)}{\partial p} = \phi^{[1]}(x, y; p) = \sum_{j=1}^{\infty} \frac{\partial a_j(p)}{\partial p}\phi_j(x, y) = \sum_{j=1}^{\infty} a_j^{[1]}(p)\phi_j(x, y). \quad (72)$$

It can be proved that $\phi^{[1]}(x, y; p)$ described above also satisfies the Laplace equation (62) and the corresponding bed condition (64).

In practice, only a finite number of terms $\phi_j(x, y)$ can be selected. Similar to the method of Rienecker and Fenton,[2] this is the only approximation made in this numerical method. Suppose $M$ terms $\phi_j(x, y)$ $(j = 1, 2, \dots, M)$ are selected. Then one has

$$\phi(x, y; p) = \sum_{j=1}^{M} a_j(p)\phi_j(x, y), \quad (73)$$

$$\phi^{[1]}(x, y; p) = \sum_{j=1}^{M} a_j^{[1]}(p)\phi_j(x, y). \quad (74)$$

For simplicity, define

$$\mathbf{a}(p) = \{a_j(p)\}, \qquad \mathbf{a}^{[1]}(p) = \{a_j^{[1]}(p)\} \qquad (j = 1, 2, \dots, M).$$

Substituting (73) and (74) in (63) one can obtain the equation for the first-order process derivatives $\mathbf{a}^{[1]}(p)$ as

$$\sum_{j=1}^{M} [b_1(x, y, \mathbf{a}; p)(\phi_j)_x + b_2(x, y, \mathbf{a}; p)(\phi_j)_y + b_3(x, y, \mathbf{a}; p)(\phi_j)_{xx} + b_4(x, y, \mathbf{a}; p)(\phi_j)_{xy}]a_j^{[1]}$$

$$= T(x, y, \mathbf{a}; p) \quad \text{on } y = \eta(x; p), \quad (75)$$

where

$$(\phi_j)_x = \frac{(jk)\cosh(jky)\cos(jkx)}{\cosh(jkD)}, \quad (76)$$

$$(\phi_j)_y = \frac{(jk)\sinh(jky)\sin(jkx)}{\cosh(jkD)}, \quad (77)$$

$$(\phi_j)_{xx} = \frac{-(jk)^2\cosh(jky)\sin(jkx)}{\cosh(jkD)}, \quad (78)$$

$$(\phi_j)_{xy} = \frac{(jk)^2\sinh(jky)\cos(jkx)}{\cosh(jkD)}. \quad (79)$$

Select $\mathbf{a}_0 = \{\bar{a}_j\}$ $(j = 1, 2, \ldots, M)$ as the initial value of $\mathbf{a}(p)$, where

$$\bar{a}_1 = \frac{H_{w0}}{2} \sqrt{\left(\frac{g}{k}\right)} \cosh(kD),$$ (80)

$$\bar{a}_j = 0 \quad \text{for } j = 2, 3, \ldots, M.$$ (81)

Then one obtains the initial velocity potential $\phi_0(x, y)$ as

$$\phi_0(x, y) = \phi(x, y; 0) = \bar{a}_1 \phi_1(x, y).$$ (82)

The initial wave elevation $\eta_0(x)$ can be given in the form

$$\eta_0(x) = \frac{H_{w0}}{2} \cos(kx).$$ (83)

In order to obtain the value of $\mathbf{a}^{[1]}(p) = \{a_j^{[1]}(p)\}$ $(j = 1, 2, \ldots, M)$ from the linear equation (75), $M$ points $(x_i, y_i)$ $(i = 1, 2, \ldots, M)$ on the free surface $y = \eta(x; p)$ must be given. Owing to the symmetry, one can distribute these $M$ points on the region $[L_w/2, L_w]$ as

$$x_j = \frac{L_w}{2} \left[ 1 + \left(\frac{j}{M-1}\right)^\alpha \right], \qquad y_j = \frac{H_{w0}}{2} \cos(kx_j) \qquad (j = 0, 1, 2, \ldots, M - 1),$$

where $L_w$ is the wave length. Here the parameter $\alpha$ is used to determine the distribution form of the $M$ points. When $\alpha = 1$, the $M$ points are equally spaced in the horizontal direction. When $\alpha < 1$, the points are spaced coarsely near the crest $(x = L_w)$, similar to the Vanden-Broeck–Schwartz method.[1]

Then, according to (75), a linear equation with $M$ unknown quantities $a_j^{[1]}(p_m)$ $(j = 1, 2, \ldots, M)$ for $p_m = m\Delta p$ can be obtained:

$$\mathbf{E}(p_m)\mathbf{a}^{[1]}(p_m) = \mathbf{t}(p_m),$$ (84)

where

$$\mathbf{t}(p_m) = \{ T_i(x_i, y_i, \mathbf{a}(p_m); p_m)\} \quad (i = 1, 2, \ldots, M),$$ (85)

$$\mathbf{E}(p_m) = [e_{ij}(x, y; p_m)],$$ (86)

with

$$e_{ij}(x_i, y_i; p_m) = b_1[x_i, y_i, \mathbf{a}(p_m); p_m][\phi_j(x_i, y_i)]_x + b_2[x_i, y_i, \mathbf{a}(p_m); p_m][\phi_j(x_i, y_i)]_y$$
$$+ b_3[x_i, y_i, \mathbf{a}(p_m); p_m][\phi_j(x_i, y_i)]_{xx} + b_4[x_i, y_i, \mathbf{a}(p_m); p_m][\phi_j(x_i, y_i)]_{xy}$$
$$(i, j = 1, 2, \ldots, M).$$ (87)

If $\mathbf{a}^{[1]}(p_m)$ is known, one has

$$\mathbf{a}(p_m + \Delta p) = \mathbf{a}(p_m) + \mathbf{a}^{[1]}(p_m)\Delta p \quad \text{for } p_m + \Delta p \leqslant 1,$$ (88)

$$\phi^{[1]}[x, y, \mathbf{a}^{[1]}(p_m); p_m] = \sum_{j=1}^{M} a_j^{[1]}(p_m)\phi_j(x, y) \quad \text{for } p_m + \Delta p \leqslant 1.$$ (89)

Then $\eta^{[1]}(x_i; p_m)$ can be obtained as

$$\eta^{[1]}(x_i; p_m) = \frac{f_1'(p_m)H[x_i, y_i, \mathbf{a}(p_m)] + f_2'(p_m)\eta_0(x_i) + f_1(p_m)\partial H[x_i, y_i, \mathbf{a}(p_m)]/\partial p}{1 - f_1(p_m)\partial H[x_i, y_i, \mathbf{a}(p_m)]/\partial y},$$ (90)

where

$$\frac{\partial H[x_i, y_i, \mathbf{a}(p_m)]}{\partial p} = \frac{\partial H[x_i, y_i, \mathbf{a}(p_m)]}{\partial \phi_x} \phi_x^{[1]}[x_i, y_i, \mathbf{a}^{[1]}(p_m)]$$

$$+ \frac{\partial H[x_i, y_i, \mathbf{a}(p_m)]}{\partial \phi_y} \phi_y^{[1]}[x_i, y_i, \mathbf{a}^{[1]}(p_m)], \tag{91}$$

$$\frac{\partial H[x_i, y_i, \mathbf{a}(p_m)]}{\partial y} = \frac{\partial H[x_i, y_i, \mathbf{a}(p_m)]}{\partial \phi_x} \phi_{xy}[x_i, y_i, \mathbf{a}(p_m)]$$

$$- \frac{\partial H[x_i, y_i, \mathbf{a}(p_m)]}{\partial \phi_y} \phi_{xx}[x_i, y_i, \mathbf{a}(p_m)]. \tag{92}$$

Thus the new free surface elevation can be obtained as

$$\eta(x_i; p_m + \Delta p) = \eta(x_i; p_m) + \eta^{[1]}(x_i; p_m)\Delta p \quad \text{for} \quad p_m + \Delta p \leqslant 1. \tag{93}$$

Therefore from the above method one can obtain the values of $\mathbf{a}(\Delta p)$ and $\eta(x; \Delta p)$ if the initial wave elevation $\eta_0(x)$ and initial velocity potential $\phi_0(x, y)$ are given by selecting freely the initial wave height $H_{w0}$. In the same way, from $\mathbf{a}(\Delta p)$ and $\eta(x; \Delta p)$ one can obtain $\mathbf{a}(2\Delta p)$ and $\eta(x; 2\Delta p)$, etc. If $\mathbf{a}(n_p\Delta p)$ and $\eta(x; n_p\Delta p)$ are known for $\Delta p = 1/n_p$, then the solution of the original equations (6)–(9) has been obtained.

The error in this numerical method is due to two reasons. One is that a finite number of terms $\phi_j(x, y)$ $(j = 1, 2, \ldots, M)$ have been used. On the other hand, the expressions (88) and (93) are obtained by dispersing the integral relations (26) and (27). This means that the process region $p \in [0, 1]$ is discretized.

It is very interesting that the method described above can successfully avoid the use of iterative methods which have traditionally been applied to solve non-linear problems. Therefore it is possible for the above method to overcome the disadvantages of iterative methods.

For the above numerical method, the wave length $L_w$, water depth $D$ and wave velocity $C$ have to be known quantities and a selected initial wave height $H_{w0}$ must be given. One can select the wave depth $D$ and gravitational acceleration $g$ so as to make all quantities non-dimensional. Thus only the ratio of wave length to water depth, $L_w/D$, and the non-dimensional wave velocity $kC^2/g$ have been known quantities and an initial ratio of wave height to wave length, $H_{w0}/L_w$, must be selected.

### 4.2. Numerical results

The method described in this paper is based on the velocity potential function $\phi(x, y)$, whereas the methods of Vanden-Broeck and Schwartz[1] and Rienecker and Fenton[2] were based on the streamfunction $\psi(x, y)$. Using the streamfunction $\psi(x, y)$, the kinematic condition of the free surface can be represented simply as

$$\psi(x, y) = -Q \quad \text{on} \quad y = \eta(x).$$

Therefore all the results of other authors were given under a constant value of $\exp(-kQ/C)$, e.g. 0·5 or 0·9. $\text{Exp}(-kQ/C)$ is a function of $L_w/D$ and $kC^2/g$. For constant $\exp(-kQ/C)$ the value of $L_w/D$ is a function of $kC^2/g$. However, the method described in this paper does not directly use $-Q$, the value of the streamfunction on the wave free surface. Instead, the non-dimensional wave velocity $kC^2/g$ is a known quantity and the wave height-to-length ratio $H_w/L_w$ is the numerical result obtained by means of the FPM. This is very different from the methods of

other authors. Therefore it is rather more difficult to compare directly the results obtained by the FPM with those of other authors. In spite of this, approximate comparisons are given in order to examine our numerical method, although a direct comparison could not be made.

The solutions of the original equations (6)–(9) for gravity waves in shallow water are obtained indirectly by solving $n_p$ linear algebraic equations ($n_p \geqslant 1$, $\Delta p = 1/n_p$). Therefore the simplest way to examine our new numerical method is to inspect whether or not the two boundary conditions (7) and (9) have been satisfied. For simplicity, let

$$e_1 = \max \left| \frac{R[\phi(x, \eta_f)]}{\sqrt{(Dg^3)}} \right| \tag{94}$$

and

$$e_2 = \max \left| \frac{\eta_f - H[\phi(x, \eta_f)]}{D} \right| \tag{95}$$

denote the maximum non-dimensional errors of boundary conditions (7) and (9) respectively, where $\eta_f$ is the numerical result for the wave elevation.

Another way to test our numerical method is to examine whether or not the values of the streamfunction $\psi(x, y)$ on the wave elevation are the same. According to (73), the horizontal velocity $u(x, y)$ can be represented as

$$u = \frac{\partial \phi}{\partial x} = \sum_{j=1}^{M} \frac{a_j(1)(jk)\cosh(jky)\cos(jkx)}{\cosh(jkD)}. \tag{96}$$

In the frame of reference moving at the same velocity $C$ as the wave, the horizontal velocity $u'$ is

$$u' = u - C.$$

Similar to other authors, one can suppose

$$\psi(x, 0) = 0.$$

Thus one can obtain the value $-Q$ of the streamfunction $\psi(x, y)$ on the wave elevation $\eta_f(x)$ as

$$
\begin{aligned}
-Q_j &= \int_0^{\eta_f(x_j)} \frac{\partial \psi}{\partial y} \, dy \\
&= \int_0^{\eta_f(x_j)} u' \, dy \\
&= \int_0^{\eta_f(x_j)} \left( \sum_{i=1}^{M} \frac{a_i(1)(ik)\cosh(iky)\cos(ikx_j)}{\cosh(ikD)} - C \right) dy \\
&= \sum_{i=1}^{M} \frac{a_i(1)\sinh[ik\eta_f(x_j)]\cos(ikx_j)}{\cosh(ikD)} - C\eta_f(x_j).
\end{aligned}
\tag{97}
$$

Define

$$-\bar{Q} = \frac{1}{M} \left( \sum_{j=1}^{M} -Q_j \right),$$

$$\lambda_{max} = \max \left| \frac{Q_j - \bar{Q}}{\bar{Q}} \right| \quad \forall j \leqslant M.$$

Similar to the method of Rienecker and Fenton,[2] the $\cosh(jkD)$ term was introduced in expression (71) in order to avoid the undesirable numerical errors caused by the large value of $j$. When the number of points $M$ is large, it is suggested to use double-precision data in the computer programme. We used VAX G-Floating data in our programme, which was executed on a VAX/VMS computer. It seems particularly necessary to use G-Floating data for very long waves and for waves near the highest.

*4.2.1. Convergence of finite process method.* Since it is a new kind of numerical method, the convergence of the FPM is first examined. We select

$$f_1(p) = p^{1 \cdot 1}, \qquad f_2(p) = (1 - p)^{1 \cdot 1}$$

Example 1

Select $L_w/D = 9$, $kC^2/g = 0.7$, $H_{w0}/D = 0.5$, $M = 32$ and $\alpha = 0.9$. The numerical results are given in Table I.

Example 2

Select $L_w/D = 20$, $kC^2/g = 0.35$, $H_{w0}/D = 0.5$, $M = 32$ and $\alpha = 0.65$. The numerical results are given in Table II.

Example 3

Select $L_w/D = 40$, $kC^2/g = 0.20$, $H_{w0}/D = 0.5$, $M = 32$ and $\alpha = 0.9$. The numerical results are given in Table III.

Example 4

Select $L_w/D = 60$, $kC^2/g = 0.13$, $H_{w0}/D = 0.5$, $M = 64$ and $\alpha = 0.95$. The numerical results are given in Table IV.

The wave with $L_w/D = 9$ is a moderately long wave. The waves corresponding to $L_w/D = 20$, $L_w/D = 40$ and especially $L_w/D = 60$ are very long waves. From Tables I–IV one can see that

Table I. Convergence of finite process method ($L_w/D = 9$)

| $\Delta p$ | 1/10 | 1/20 | 1/50 | 1/100 | 1/200 | 1/500 |
|---|---|---|---|---|---|---|
| $H_w/D$ | 0·4913933 | 0·4976689 | 0·5045102 | 0·5072117 | 0·5086361 | 0·5094969 |
| $\exp(-kQ/C)$ | 0·5108812 | 0·5104637 | 0·5102284 | 0·5101570 | 0·5101228 | 0·5101022 |
| $\lambda_{max}$ | $3·4803 \times 10^{-2}$ | $1·6840 \times 10^{-2}$ | $6·6904 \times 10^{-3}$ | $3·3450 \times 10^{-3}$ | $1·6768 \times 10^{-3}$ | $6·7507 \times 10^{-4}$ |
| $e_1$ | $1·8268 \times 10^{-2}$ | $9·4735 \times 10^{-3}$ | $3·8581 \times 10^{-3}$ | $1·9396 \times 10^{-3}$ | $9·7242 \times 10^{-4}$ | $3·8592 \times 10^{-4}$ |
| $e_2$ | $3·6731 \times 10^{-2}$ | $1·8009 \times 10^{-2}$ | $7·2612 \times 10^{-3}$ | $3·6448 \times 10^{-3}$ | $1·8296 \times 10^{-3}$ | $7·3320 \times 10^{-4}$ |

Table II. Convergence of finite process method ($L_w/D = 20$)

| $\Delta p$ | 1/10 | 1/20 | 1/50 | 1/100 | 1/200 | 1/500 |
|---|---|---|---|---|---|---|
| $H_w/D$ | 0·2970015 | 0·3122617 | 0·3215379 | 0·3238492 | 0·3262390 | 0·3271250 |
| $\exp(-kQ/C)$ | 0·7337657 | 0·7335614 | 0·7334444 | 0·7334083 | 0·7333886 | 0·7333791 |
| $\lambda_{max}$ | $2·0893 \times 10^{-2}$ | $1·0688 \times 10^{-2}$ | $4·3367 \times 10^{-3}$ | $2·1737 \times 10^{-3}$ | $1·0966 \times 10^{-3}$ | $4·4570 \times 10^{-4}$ |
| $e_1$ | $1·3901 \times 10^{-3}$ | $7·4959 \times 10^{-4}$ | $3·1629 \times 10^{-4}$ | $1·6736 \times 10^{-4}$ | $8·4218 \times 10^{-5}$ | $3·3900 \times 10^{-5}$ |
| $e_2$ | $2·4917 \times 10^{-2}$ | $1·2999 \times 10^{-2}$ | $5·1431 \times 10^{-3}$ | $2·5839 \times 10^{-3}$ | $1·2946 \times 10^{-3}$ | $5·1874 \times 10^{-4}$ |

Table III. Convergence of finite process method ($L_w/D = 40$)

| $\Delta p$ | 1/10 | 1/20 | 1/50 | 1/100 | 1/200 | 1/500 |
|---|---|---|---|---|---|---|
| $H_w/D$ | 0·3723232 | 0·3911771 | 0·4029229 | 0·4069286 | 0·4089504 | 0·4101699 |
| $\exp(-kQ/C)$ | 0·8561234 | 0·8560977 | 0·8560824 | 0·8560773 | 0·8560748 | 0·8560732 |
| $\lambda_{max}$ | $3\cdot2188 \times 10^{-2}$ | $1\cdot6403 \times 10^{-2}$ | $6\cdot6533 \times 10^{-3}$ | $3\cdot3443 \times 10^{-3}$ | $1\cdot6775 \times 10^{-3}$ | $6\cdot7311 \times 10^{-4}$ |
| $e_1$ | $3\cdot6030 \times 10^{-3}$ | $1\cdot9052 \times 10^{-3}$ | $7\cdot8991 \times 10^{-4}$ | $3\cdot9985 \times 10^{-4}$ | $2\cdot0117 \times 10^{-4}$ | $8\cdot0770 \times 10^{-5}$ |
| $e_2$ | $3\cdot5535 \times 10^{-2}$ | $1\cdot8104 \times 10^{-2}$ | $7\cdot3405 \times 10^{-3}$ | $3\cdot6884 \times 10^{-3}$ | $1\cdot8489 \times 10^{-3}$ | $7\cdot4071 \times 10^{-4}$ |

Table IV. Convergence of finite process method ($L_w/D = 60$)

| $\Delta p$ | 1/10 | 1/20 | 1/50 | 1/100 | 1/200 | 1/500 |
|---|---|---|---|---|---|---|
| $H_w/D$ | 0·2927140 | 0·3074407 | 0·3165027 | 0·3195666 | 0·3211073 | 0·3220346 |
| $\exp(-kQ/C)$ | 0·9010935 | 0·9010919 | 0·9010918 | 0·9010918 | 0·9010919 | 0·9010920 |
| $\lambda_{max}$ | $2\cdot5770 \times 10^{-2}$ | $1\cdot3133 \times 10^{-2}$ | $5\cdot5239 \times 10^{-3}$ | $2\cdot6744 \times 10^{-3}$ | $1\cdot3398 \times 10^{-3}$ | $5\cdot3588 \times 10^{-4}$ |
| $e_1$ | $2\cdot4439 \times 10^{-3}$ | $1\cdot3125 \times 10^{-3}$ | $5\cdot5005 \times 10^{-4}$ | $2\cdot7951 \times 10^{-4}$ | $1\cdot4091 \times 10^{-4}$ | $5\cdot6643 \times 10^{-5}$ |
| $e_2$ | $2\cdot7968 \times 10^{-2}$ | $1\cdot4269 \times 10^{-2}$ | $5\cdot7874 \times 10^{-3}$ | $2\cdot9081 \times 10^{-3}$ | $1\cdot4578 \times 10^{-3}$ | $5\cdot8401 \times 10^{-4}$ |

there is convergence for each case. Although the two boundary conditions (7) and (9) are not solved directly, their maximum non-dimensional errors $e_1$ and $e_2$ decrease as $\Delta p$ is decreased. One could speculate that $e_1$ and $e_2$ will tend to zero if $\Delta p$ tends to zero. In the same way, $\lambda_{max}$ also decreases, which means that the streamfunction value $-Q$ on the wave elevation tends to the same. It should be emphasized that this streamfunction value $-Q$ is also obtained indirectly.

Although the FPM also needs an initial solution, it is insensitive to the initial values. For Example 4, $L_w/D = 60$, corresponding to a very long wave, the wave elevation is very different from that of the deep water wave which has been applied as the initial wave elevation. Figure 1 shows comparison of the initial deep water wave with the numerical long wave obtained by the FPM for $\Delta p = 0\cdot02$ (Example 4). With the iterative methods used by other authors, it was necessary sometimes to use the converged solutions obtained for lower waves as the initial approximation.

By using the FPM, iterative methods can be successfully avoided. The FPM is insensitive not only to the selected initial solution but also to the number of unknowns. Therefore the disadvantages and limitations of iterative methods can be overcome. The disadvantage of the FPM is that it needs more CPU time. This is mainly because the precision is approximately directly proportional to the CPU time for the FPM. From Tables I–IV one can see that the errors $e_1$ and $e_2$ are approximately directly proportional to $\Delta p$.

*4.2.2. Comparison with the results of other authors.* The numerical results of Rienecker and Fenton[2] agreed very well with the results of Cokelet[5] and Vanden-Broeck and Schwartz,[1] and even agreed with the experimental results given by Le Méhauté.[8] It does not seem necessary to make a further detailed comparison of all of these, for two reasons. One is that the purpose of this paper is mainly to introduce and examine a new kind of numerical method which can avoid using iterative methods and can be used to solve 3D non-linear gravity wave problems, e.g. the wave resistance of a ship moving in deep or shallow water. Another reason, which is perhaps the main one, is that the method described in this paper is based on the velocity potential function and therefore it is very difficult to compare the results directly with those of other authors.

In spite of this, some approximate comparisons have been made and are given in Table V.

For a particular value of $kC^2/g$, a large number of computations are needed in order to find a value of $L_w/D$ such that $\exp(-kQ/C) \approx 0\cdot5$. This seems especially difficult for waves near the
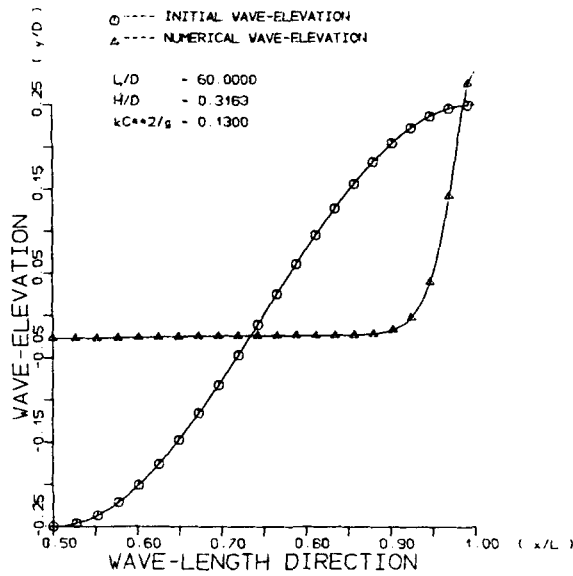
Figure 1. Long wave with $L_w/D = 60$ for $\Delta p = 0.02$

Table V. Comparisons with results of other authors

| | Results of present method (FPM) | | | Results of other authors $(\exp(-kQ/C) = 0.5)$ | | | |
| $L_w/D$ | $\exp(-kQ/C)$ | $kC^2/g$ | $H_w/D$ | $H_w/D$ | $kC^2/g$ | | |
| | | | | | Reference 5 | Reference 2 | Reference 1 |
| 9·0158 | 0·5000197 | 0·615059 | 0·1720615 | 0·1729974 | 0·615059 | 0·615059 | Not given |
| 8·9776 | 0·5006100 | 0·631112 | 0·2528818 | 0·252630 | 0·631112 | 0·631112 | Not given |
| 8·8670 | 0·5004126 | 0·666501 | 0·3791625 | 0·3802645 | 0·666501 | 0·666501 | 0·666501 |
| 8·7634 | 0·5004327 | 0·706443 | 0·4898292 | 0·4944549 | 0·706443 | 0·706443 | 0·706443 |
| 8·6900 | 0·50076340 | 0·748230 | 0·5959385 | 0·602447 | 0·748230 | 0·748230 | 0·748230 |
| 8·6800 | 0·5097155 | 0·764403 | 0·6497747 | 0·651251 | 0·764403 | 0·764403 | 0·764403 |
| 8·6795 | 0·5008378 | 0·767748 | 0·6699720 | 0·672143 | 0·767748 | 0·767748 | 0·767748 |

highest. The corresponding values of $L_w/D$ are also given in Table V. Although it is not possible to make a direct comparison, from the results of Table V it would seem reasonable to believe that the present method could give results which agree well with those of References 1, 2 and 5.

One can compare the present method with the method of Rienecker and Fenton.[2] Similar to other authors, their method was based on the streamfunction and Newton's iterative method was used to solve the corresponding non-linear algebraic equation. If $M$ points on the wave elevation are selected, a non-linear algebraic equation with $2M + 5$ unknowns must be solved; however, with the present method, only $n_p$ linear equations with $M$ unknown quantities need be solved, because at each step $p_m = m\Delta p$ the wave elevation $y = \eta(x; p_m)$ is known. Unlike the methods based on iterative techniques, the FPM is insensitive to the initial solution and the number of unknowns. However, much more CPU time is needed. This is perhaps the greatest disadvantage of the FPM.

## 5. CONCLUSIONS

In this paper a typical non-linear problem, namely 2D steep gravity waves in shallow water, was used to introduce and examine a new kind of numerical method called the finite process method.

The FPM can successfully avoid using iterative methods and therefore can overcome the disadvantages and limitations of such methods. In comparison with iterative methods, the FPM is insensitive to the selection of the initial solution and the number of the unknowns. Even for waves near the highest or for very long waves, e.g. waves with $L_w/D = 120$, a converged numerical result can be obtained by using the deep water wave as the initial solution. With the method of Rienecker and Fenton[2] it was necessary in this situation to extrapolate to the initial approximation from converged solutions for lower waves.

Although it is rather more difficult to compare the results of the present method directly with those of other authors, from the approximate comparisons given in Table V one could have reason to believe that the FPM could indeed be used for the numerical solution of 2D steep gravity waves in shallow water. It is interesting that, in contrast with the methods of other authors, the present method is based on the velocity potential function and therefore can be applied to solve 3D non-linear gravity waves by using the Hess–Smith method, e.g. the wave resistance of a ship moving in deep or shallow water, which will be discussed in Part 2 of this series of the author.

In this paper the concepts of *process* and *process derivatives* were used. The process is a kind of continuous mapping which connects the selected initial solution with the solutions of what one wants to solve. It is interesting that the equations for the first-order process derivatives, called the first-order process equations, are linear. By means of this property of process, a non-linear problem (e.g. a non-linear algebraic equation, a set of non-linear algebraic equations, a differential or partial differential equation, etc.) can be transformed into a set of linear problems, because it is only needed for the FPM to solve $n_p$ linear equations ($n_p \geq 1$, $\Delta p = 1/n_p$). In other words, a non-linear problem can be approximated by a finite number of linear problems, or perhaps one could say that a non-linear problem can be discretized into $n_p$ linear problems ($n_p \geq 1$). Also, the finer the discretization is, i.e. the smaller $\Delta p$ is, the more exact is this kind of approximation. This is similar to the finite element method and also to the finite difference method if we regard each corresponding linear problem as an element of a non-linear problem. It is well known that a real function can generally be approximated by a finite number of terms of a Fourier series. One can see that the FPM is similar if we regard the non-linear problem as the function and the linear problems as the Fourier series.

The disadvantage of the FPM is that more CPU time is needed in order to obtain a sufficiently accurate result. This is mainly because its precision is approximately directly proportional to the required CPU time. Thus a more exact result means a higher price. For engineering problems it is suggested not to use at first a very small value of $\Delta p$.

## APPENDIX: NOMENCLATURE

| | |
|---|---|
| $C$ | wave velocity |
| $D$ | average water depth |
| $E$ | matrix of algebraic equations |

$e_{ij}$                           element of $\mathbf{E}$

$e_1$                            maximum non-dimensional error of boundary condition (7)

$e_2$                            maximum non-dimensional error of boundary condition (9)

$f_1(p)$                          first type of process function

$f_2(p)$                          second type of process function

$g$                             gravitational acceleration

$k$                             wave number

$H_w$                            wave height

$H_{w0}$                          selected initial wave height

$H[\phi(x, y)]$                      auxiliary function defined in expression (5)

$L_w$                            wave length

$M$                             number of selected points on the wave elevation

$p$                             process-independent variable

$-Q$                            value of streamfunction $\psi(x, y)$ on free surface

$-\bar{Q}$                            average value of streamfunction on free surface

$R[\phi(x, y)]$                      auxiliary function defined in expression (4)

$S[\phi(x, y; p); p]$                   auxiliary function defined in expression (35)

$T[\phi(x, y; p); p]$                   auxiliary function defined in expression (37)

$(u, v)$                          velocity components

$(x, y)$                          co-ordinates

$\psi(x, y)$                         streamfunction

$\phi(x, y)$                         velocity potential function

$\phi_0(x, y)$                        initial velocity potential function

$\phi_f(x, y)$                        final velocity potential function

$\phi(x, y; p)$                       mapping of $\phi(x, y)$, called the zero-order process for $\phi(x, y)$

$\phi^{[1]}(x, y; p)$                    first-order process derivatives for $\phi(x, y; p)$

$\eta(x)$                          wave elevation

$\eta_0(x)$                         selected initial wave elevation

$\eta_f(x)$                         final wave elevation

$\eta(x; p)$                        mapping of $\eta(x)$, called the zero-order process for $\eta(x)$

$\eta^{[1]}(x; p)$                     first-order process derivatives for $\eta(x; p)$

$\alpha$                            parameter determining the distribution of discretized points

$\lambda_{max}$                         maximum non-dimensional error of $-Q$ on wave elevation

$\Omega$                            region of fluid

$\Omega_0$                           initial region of fluid

## REFERENCES

1. J.-M. Vanden-Broeck and L. W. Schwartz, 'Numerical computation of steep gravity waves in shallow water', *Phys. Fluids*, **22**, 1868–1871 (1979).
2. M. M. Rienecker and J. D. Fenton, 'A Fourier approximation method for steady water waves', *J. Fluid Mech.*, **104**, 119–137 (1981).
3. J. R. Chaplin, 'Developments of stream-function wave theory', *Coastal Eng.*, **3**, 179–205 (1980).
4. J. E. Chappelear, 'Direct numerical calculation of wave properties', *J. Geophys. Res.*, **66**, 501–508 (1961).
5. E. D. Cokelet, 'Steep gravity waves in water of arbitrary uniform depth', *Phil. Trans. R. Soc. A*, **286**, 183–230 (1977).
6. R. G. Dean, 'Stream function representation of nonlinear ocean waves', *J. Geophys. Res.*, **70**, 4561–4572 (1965).
7. P. S. Eagleson, 'Properties of shoaling waves by theory and experiment', *Trans. Am. Geophys. Union*, **37**, 565–572 (1956).
8. B. Le Méhauté, D. Divoky and A. Lin, 'Shallow water waves: a comparison of theories and experiments', *Proc. 11th Conf. on Coastal Engineering*, **1**, 86–107 (1968).

9. M. S. Longuet-Higgins, 'Integral properties of periodic gravity waves of finite amplitude', *Proc. R. Soc. A*, **342**, 157–174 (1975).
10. G. G. Stokes, 'On the theory of oscillatory waves', *Trans. Camb. Phil. Soc.*, **8**, 441–445 (1847).
11. G. G. Stokes, 'On the highest waves of uniform propagation', *Proc. Camb. Phil. Soc.*, **4**, 361–365 (1883).
12. J. D. Fenton, 'A high-order cnoidal wave theory', *J. Fluid Mech.*, **94**, 129–161 (1979).
13. L. W. Schwartz, 'Computer extension and analytical continuation of Stokes expansion for gravity waves', **62**, pt. 3, 553–578 (1974).
14. J. W. Macarthur, 'Robust semidirect finite difference methods for solving the Navier–Stokes and energy equations', *Int. j. numer. methods fluids*, **9**, 325–340 (1989).